Representation theory talk, Sep 21, 2017, 3.30 pm

Remarks on SH^c of Schiffmann + Vasserot

"Cherednik algebras, W-algebras and equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$", arXiv: 1202.2756

Arbesfeld + Schiffmann, 'A presentation of the deformed $W_{1+\infty}$ algebra', arXiv: 1209.0429

Kanno, Matsuo + Zhang, 1207.5658, 1306.1523,

Nakamura, Okazawa + Matsuo, 1411.4222

Fukuda, Nakamura, Matsuo + Zhu, 1509.01000

Bourgine, Matsuo + Zhang, 1512.02492

Hong Zhang, PhD thesis (available online)

Bourgine, 1407.8341

Many origins of SH^c

DAHA

Schiffmann + Vasserot motivated by AGT

Elliptic algebras

Feigin, Feigin, Jimbo, Miwa, Mukhin motivated by AGT

String th, 2d cft $W_{1+\infty}$

Miki 2007

qu continuous gl_∞

$[q,t]$-deformation of $W_{1+\infty}$

SH^c, a β-deformation of $W_{1+\infty}$.

quantum toroidal gl_1

these are supposed to be isomorphic. I know that certain representations are the same, in the correct limits.

I have left out many earlier works:

- Odenskii and Feigin 1989, ..
- Ding - Iohara 1997
- Awata, Feigin, Shiraishi et al, ~ 2010

SH^c $\quad D_{m,n}$

$m \in \mathbb{Z}$ $\quad n \in \mathbb{N}$ $\leftarrow 0, 1, 2, \ldots$

the degree $\quad$ the order

The comm rel's for $m = -1, 0, +1$ generators are the

$$[D_{0,n_1}, D_{1,n_2}] = D_{1,n_1+n_2-1}, \quad n_1 = 1, 2, \ldots$$

$$[D_{0,n_1}, D_{-1,n_2}] = -D_{-1,n_1+n_2-1}, \quad n_1 = 1, 2, \ldots$$

$$[D_{-1,n_1}, D_{+1,n_2}] = E_{n_1+n_2}, \quad [n_1, n_2] = 1, 2, \ldots$$

$$[D_{0,n_1}, D_{0,n_2}] = 0, \quad [n_1, n_2] = 0, 1, 2, \ldots$$

E_i is a non-linear combination of $D_{0,n}$, determined by

$$1 + \beta' \sum_{i \geq 0} E_i x^{i+1} = \exp \sum_{j \geq 0} (-)^{j+1} c_j \pi_j(x)$$

$$\beta' = 1 - \beta$$

$$\exp \sum_{j' \geq 0} D_{0,j'+1} \omega_{j'}(x)$$

$$\pi_{j'}(x) = x^j G_j [1 - \beta' x]$$

$$\omega_{j'}(x) = x^j \sum \left[G_j(1 - \eta x) - G_j(1 + \eta x) \right]$$

$$\eta = 1, -\beta, -\beta'$$

$$G_0[x] = -\ln x, \quad G_j(x) = [x^{-j} - 1]/j, \quad j = 1, 2, \ldots$$

The parameters c_j, $j = 0, 1, 2, \ldots$, are central char

The first few E_j's are

$E_0 = c_0$, $E_1 = -c_1 - \frac{1}{2} c_0 c_0' \beta'$,

$E_2 = c_2 - c_0' c_1 \beta' - \frac{1}{6} c_0 c_0' c_0'' \beta'^2 \quad + 2\beta D_{0,1}$

$E_3 = 6\beta D_{0,2} + 2 c_0 \beta \beta' D_{0,1} + \ldots$

$c_j' = 1 - c_j$

The remaining generators are defined recursively by,

does not contain $D_{0,n}$
(I expect will depend on c_3)

$c_j'' = 2 - c_j$

$$D_{m+1,0} = \frac{1}{m} [D_{1,1}, D_{m,0}]$$

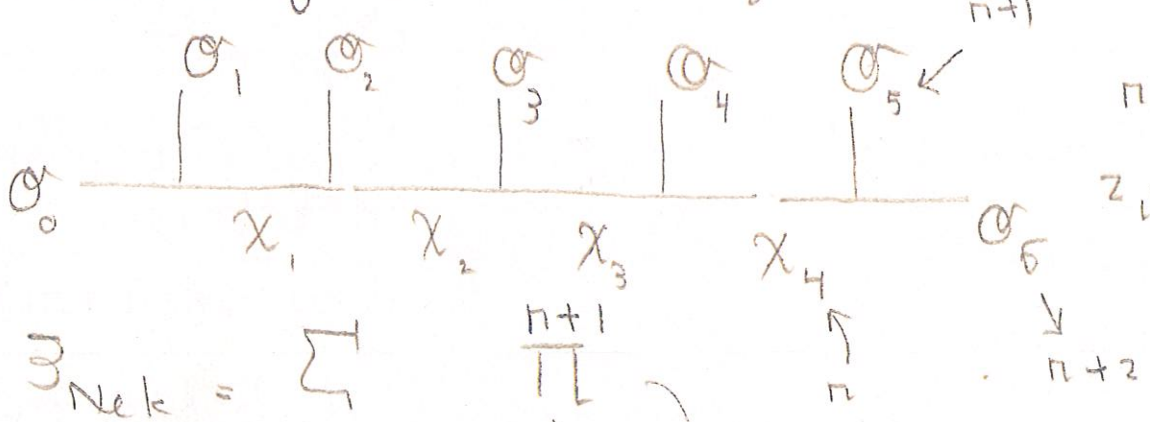
$$D_{-m-1,0} = -\frac{1}{m} [D_{-1,1}, D_{-m,0}]$$

$$D_{m,n} = [D_{0,n+1}, D_{m,0}]$$

$$D_{-m,n} = -[D_{0,n+1}, D_{-m,0}],$$ for $m = 1, 2, \ldots$,
$n = 0, 1, 2, \ldots$

A few words on AGT The main problem in exactly-solved quantum system is to compute the correlation functions. In 2d conformal field theories, the corr fun's split into conf blocks, and the problem now is to compute the conforma blocks. This turns out to be a difficult problem and was solved in special cases only. In 2009, Alday, Gaiotto and Tachikawa noticed that certain (known) conformal blocks are equal to basic objects in 4D $N=2$ super YM theories: the instanton partition functions. They conjectured that is always the case for 2D cft's based on W_2' algebra, $W_2' = W_2 \times U[1]$. The corresponding instanton partition functions were computed by Nekrasov ~ 2002/3. The AGT conjecture was (partially) extended to objects based on W_n'. There are proofs in the W_2' case by Alba, Fateev, Litvinov and Tarnopolskyi (2010). The idea of Schiffmann and Vasserot (as well as Feigin, and probably many others) is that AGT follows from the fact that the 2d conf blocks and the 4d instanton partition functions are built from matrix elements of same vertex operator that intertwine the same infinite-dim reps of the same infinite-dim algebras, Schiffmann H found that this algebra is SH^c

A conformal block is represented schematically in terms of 'a comb diagram'



$n=4$

$z_{i+2}=\infty$

$$\mathcal{Z}_{Nek} = \sum_{\bar{Y}_1,\dots,\bar{Y}_n} \prod_{i=1}^{n+1} q_i^{|\bar{Y}_i|}\, \mathcal{Z}^{i}_{bb}\left[\bar{a}_{i-1}, \bar{Y}_{i-1} \mid \mu_i \mid, \bar{b}_i, \bar{Y}_i\right]$$

N-tuple Young diagrams for models based on W_N'

$q_i = e^{2\pi i \tau_i}$

the coupling of $U(2)_i$

In conformal field theory $q_i = \dfrac{z_i}{z_{i+1}}$

$q_0 = \dfrac{z_0}{z_1} = 0$

$z_0 = 0,\ z_{n+1} = 1,\ z_{n+2} = \infty$

$q_{n+1} = z_{n+1}/z_{n+2} = 0$

$$\mathcal{Z}_{bb}[\bar{a},\bar{v}|\mu|\bar{b},\bar{w}] = \frac{z_{num}[\bar{a},\bar{v}|\mu|\bar{b},\bar{w}]}{z_{den}[\bar{a},\bar{v}|\bar{b},\bar{w}]}$$

a normalized matrix element.

$$z_{den}[\bar{a},\bar{v}|\bar{b},\bar{w}] = \left[z_{norm}[\bar{a},\bar{v}]\, z_{norm}[\bar{b},\bar{w}] \right]^{1/2}$$

$$z_{norm}[\bar{a},\bar{v}] = z_{num}[\bar{a},\bar{v}|0|\bar{a},\bar{v}]$$

$$z_{num}[\bar{a},\bar{v}|\mu|\bar{b},\bar{w}] =$$

$$\prod_{i,j=1}^{2} \prod_{\square\in v_i} [E[a_i-b_j, v_i, w_j, \square]-\mu]$$

$$\prod_{\blacksquare\in w_j} [\epsilon_1+\epsilon_2 - E[b_j-a_i, w_j, y_i, \blacksquare]-\mu]$$

$$E[x, v_i, w_j, \square] = x + A^{+}_{\square,v_i}\,\epsilon_2 - L_{\square,w_j}\,\epsilon_1$$

Schiffmann + V showed that, for suitable choices of the parameters, SH^c generates rep's with states whose norms match z_{norm}. They showed that the norm of the Gaiotto state is the instanton partition function of pure sl_n super YM. They did not get the intertwiner.

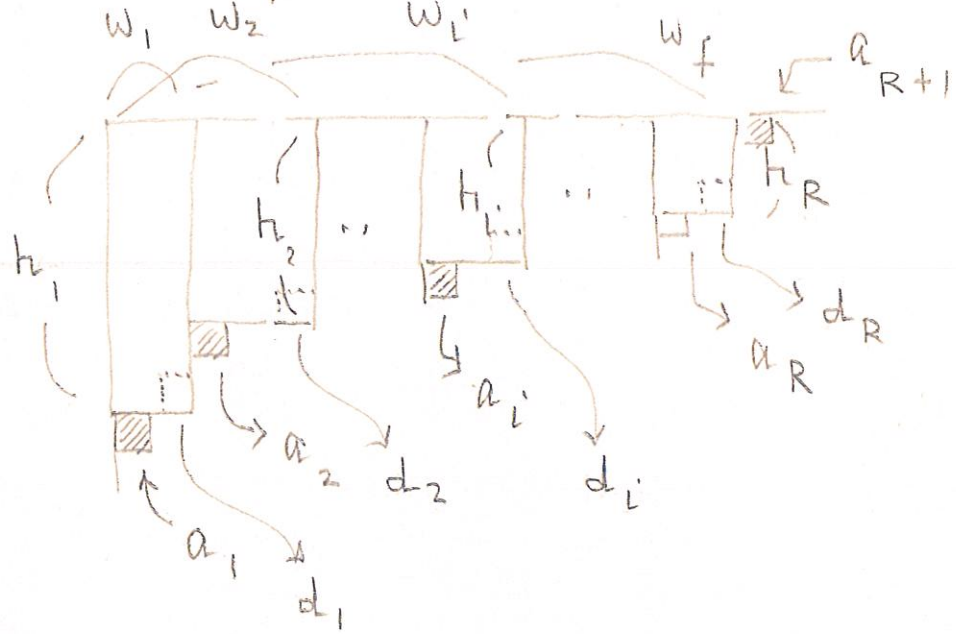
Nekrasov's deformation parameters

β in SH^c will be $-\epsilon_2/\epsilon_1$

The extra parameter in Miki's algebra is R (the M-th circle radi

In the following, I follow the proof of Matsuo et al. that shows that SH^c generates a basis that leads to the correct pure sym partition func's.

We decompose a Young diagram Y into rectangles



Starting from Y, we obtain $Y^{(+,i)}$ by adding a_i, $i=1,\dots,R$

and $Y^{(-,j)}$ by deleting d_j, $j=1,2,\dots,R+1$

We define the action of $D_{\pm,n}$ and $D_{0,n}$ on as the right-states labeled by N-Young diagrams)

$$D_{-1,n}|\bar{b},\bar{Y}\rangle = (-)^n \sum_{i=1}^{N}\sum_{j=1}^{R_i}[b_i + D_j[Y_i]]^n \Lambda_i^{(j,-)}|\bar{b},\bar{Y}^{(j,-),i}\rangle$$

$$D_{1,n}|\bar{b},\bar{Y}\rangle = (-)^n \sum_{i=1}^{N}\sum_{j=1}^{R_i+1}[b_i + A_j[Y_i]]^n \Lambda_i^{(j,+)}|\bar{b},\bar{Y}^{(j,+),i}\rangle$$

$$D_{0,n+1}|\bar{b},\bar{Y}\rangle = (-)^n \sum_{i=1}^{N}\sum_{\square\in Y_i}[b_i + c(\mu)]^n|\bar{b},\bar{Y}\rangle$$

$c[\mu] = \beta i - j$, for $\mu = (i,j)$

$A_j[Y_i] = \beta w_{j-1} - h_j - \beta'$, $j=1,\dots,R+1$

$D_j[Y_i] = \beta w_j - h_j$, $j=1,\dots,R$

$$\Lambda_i^{(j,+)}[\vec{b},\vec{y}] = \left[\prod_{i'=1}^{N}\left[\prod_{j'=1}^{R}{}' \frac{b_i - b_{i'} + A_j[y_i] - D_{j'}[y_{i'}] + \beta'}{b_i - b_{i'} + A_j[y_i] - D_{j'}[y_{i'}]}\right.\right.$$

$$\left.\left.\prod_{j'=1}^{R+1}{}' \frac{b_i - b_{i'} + A_j[y_i] - A_{j'}[y_{i'}] - \beta'}{b_i - b_{i'} + A_j[y_i] - A_{j'}[y_{i'}]}\right]\right]^{1/2}$$

adding

at position j

in diagram i

$(i',j') \neq (i,j)$

$$\Lambda_i^{(j,-)}[\vec{b},\vec{y}] = \left[\prod_{i'=1}^{N}\left[\prod_{j'=1}^{R+1} \frac{b_i - b_{i'} + D_j[y_i] - A_{j'}[y_{i'}] - \beta'}{b_i - b_{i'} + D_j[y_i] - A_{j'}[y_{i'}]}\right.\right.$$

$$\left.\left.\prod_{j'=1}^{R}{}' \frac{b_i - b_{i'} + D_j[y_i] - D_{j'}[y_{i'}] + \beta'}{b_i - b_{i'} + D_j[y_i] - D_{j'}[y_{i'}]}\right]\right]^{1/2}$$

Matsuo et al. show that these operators satisfy SH^c. They generate the correct basis that leads to pure SYM instanton partition functions.

They also show that they generate the correct intertwiners.

for $\quad c_i = \sum_{j=1}^{N} [b_j - \beta]^i$

Note that $c_0 = N$, and

the remaining c_i's are determined by the para's b_j and β. At the level of W_N, there will be only β, the central charge of Virasoro.

<u>Reduction to $W_{1+\infty}$</u>. For generic β, SH^c is a non-linear algebra. In the limit $\beta \to 1$, it reduces to the linear algebra $W_{1+\infty}$, which is the algebra of diff op's $z^n D^m$, $D = z\partial_z$.

$m = 0, 1, 2, \ldots$
$n = \ldots, -1, 0, 1, \ldots$

The operator version of $z^n D^m$ is $W[z^n D^m]$ and can be obtained by setting $\beta = 1$, $\beta' = 0$ in the SH^c relations.

<u>$W_2' = W_2 \times u(1)$ in SH^c</u>

Virasoro $u[1]$ current algebra $\sim$ Heisenberg

$$J_{\pm m} = [-\beta^{1/2}]^{-m} D_{\mp m, 0}, \quad J_0 = \beta^{-1} E_1$$

$$E_1 = -c_1 - \frac{1}{2} c_0 c_0' \beta'$$

$$L_{\pm m} = \frac{1}{m}[-\beta^{1/2}]^{-m} D_{\mp m, 1} + (1-m) c_0 \frac{\beta'}{2} J_{\pm m}$$

$$L_0 = \frac{1}{2}[L_1, L_{-1}] = D_{0,1} + \frac{1}{2\beta}\left[c_2 + c_1 c_0' \beta' + \frac{\beta'^2}{6} c_0 c_0' c_0''\right]$$

From the identifications, we obtain the usual comm rel's $[J_m, J_n] = \frac{nN}{\beta}\delta_{m+n,0}$

$$[L_m, J_n] = -n J_{m+n}$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}$$

The derivations of these formulas are far from easy since the SH^c com rel's involve generators of degree $(0, \pm 1)$ only, while J_n and L_n have degree n

$$c = \beta^{-1}[-c_0^3 \beta^{1/2} + c_0 - c_0 \beta' + c_0 \beta'^{/2}]$$

$$W_N \qquad = 1 + [N-1][1 - N(N+1)Q^2], \; Q = \beta^{1/2} - \beta^{-1/2}$$

The point is that one can use the generators $D_{m,n}$, $m \in \mathbb{Z}$, $n = 0, 1, .., N$ to cook up the W_N' generators.

<u>Remark</u> The point about these expansions is that they offer another way to represent W_N. Currently, only W_2 and W_3 (and maybe one more case) are known explicitly. W_N, $N \geq 4$ is too complicated

$\underline{W_3}$ $[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m [m^2 - 1] \delta_{m+n,0}$,

$[L_m, W_n] = (2m - n) W_{m+n}$,

$$[W_m, W_n] = (m-n) \left(\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2) \right] L_{m+n} \right)$$

$$+ \left[\frac{c}{360} \right] m (m^2 - 4)(m^2 - 1) \delta_{m+n,0}$$

$$+ \gamma (m-n) \Lambda_{m+n}$$

$$\gamma = \frac{16}{22 + 5c}, \; \Lambda_m = \sum_{p \geq -1}' L_{m-p} L_p + \sum_{p \leq -2} L_p L_{m-p}$$

$$- \frac{3}{10}(m+2)(m+3) L_m$$

Quantum toroidal gl_1 / elliptic Hall algebra

is the associative algebra generated by the current

$$x^{\pm}[z] = \sum_{m\in\mathbb{Z}}' x^{\pm}_m z^{-m}, \quad \psi^{\pm} = \sum_{\pm n \in \mathbb{N}}' \psi^{\pm}_n z^{-n}$$

$0, 1, 2, \ldots$

and the central element

$\gamma^{\pm 1/2}$, with the relations

$$\psi^{\pm}(w)\psi^{\pm}(z) = \psi^{\pm}(z)\psi^{\pm}(w),$$

$$\psi^{+}(w)\psi^{-}(z) = \frac{g[\gamma z/w]}{g[\gamma' z/w]}\psi^{-}[z]\psi^{+}[w],$$

$$\psi^{+}[w]\, x^{\pm}[z] = g[\gamma^{\mp 1/2} z/w]^{\mp 1} x^{\pm}[z]\psi^{+}[w],$$

$$\psi^{-}[w]\, x^{\pm}[z] = g[\gamma^{\mp 1/2} w/z]^{\mp 1} x^{\pm}[z]\psi^{-}[w],$$

$$[x^{+}(w), x^{-}(z)] = \frac{[1-q][1-t']}{1-qt'}\Big[\delta[\gamma' w z']\psi^{+}[\gamma^{1/2} z] - \delta[\gamma z w']\psi^{-}[\gamma^{-1/2} z]\Big]$$

$$G^{\mp}[wz']\, x^{\pm}[w]\, x^{\pm}[z] = G^{\pm}[wz']\, x^{\pm}[z]\, x^{\pm}[w]$$

where $\delta(z) = \sum_{n\in\mathbb{Z}}' z^{n}$, $g(z) = \dfrac{G^{+}(z)}{G^{-}(z)}$,

$$G^{\pm}(z) = [1-q^{\pm 1} z][1-t^{\mp} z][1-q^{\mp} t^{\pm} z]$$

This algebra depends on two complex parameters q, t and is denoted by $\mathcal{E}[q,t]$. It's rep is level m if the $\gamma^{\pm 1/2} = [t q']^{\pm m/4}$.